# ON THE STANDARD K-LOOP STRUCTURE OF POSITIVE INVERTIBLE ELEMENTS IN A $C^*$-ALGEBRA

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Abstract. We investigate the algebraic properties of the operation $a \circ b = \sqrt{a}b\sqrt{a}$ on the set of all positive invertible elements of a $C^*$-algebra $\mathcal{A}$. We show that its commutativity, associativity and distributivity are each equivalent to the commutativity of $\mathcal{A}$. We present abstract characterizations of the operation $\circ$ and a few related ones, too.

## 1. Introduction

The set of all positive definite symmetric real matrices and that of all positive definite Hermitian complex matrices equipped with the operation $A \circ B = \sqrt{A}B\sqrt{A}$ have important applications the probably best known one being due to its intimate connection to Einstein's velocity addition, an operation which plays a fundamental role in the special theory of relativity. For discussions on this connection we can refer, for example, to Chapter 10 in the monograph [14], to Theorem 5.4 (together with Lemma 5.3) in [18], to Theorem 2.6 in [16], or to [15]. We should remark that the author in [14] uses the operation $(AB^2A)^{1/2}$ in the first place but the corresponding structure is obviously isomorphic to the above one under the squaring map. The operation $\sqrt{A}B\sqrt{A}$ on the set of all positive (in matrix language positive semidefinite) Hilbert space operators bounded by the identity (which set is usually called Hilbert space effect algebra) plays a remarkable role in the mathematical foundations of quantum theory, too. In fact, in [7] Gudder and Nagy introduced this operation in relation with the quantum theory of measurements and called it sequential product. In their approach the effect $A \circ B = \sqrt{A}B\sqrt{A}$ represents the sequential measurement in which $A$ is performed first and $B$ second. In the same paper the authors investigated certain algebraic properties (especially commutativity and associativity) of that operation. Their results have motivated considerable amount of further investigations, both purely algebraic and also analytical, see, e.g., the survey paper [5] and also the articles [3], [11], [12], [20], [27] (we may also

2010 *Mathematics Subject Classification.* Primary: 46L05, 20N05. Secondary: 47B49.

*Key words and phrases.* Loop, K-loop, $C^*$-algebra, positive invertibles in a $C^*$-algebra.

This research was carried out when the second author was visiting the Department of Physics at the University of Calabria, Italy which visit was supported by INFN (Istituto Nazionale Fisica Nucleare). The research of the same author was also supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences and by the Hungarian Scientific Research Fund (OTKA) Reg.No. K81166 NK81402.

refer to Section 2.8 in the second author's book [21] where he described the structure of sequential automorphisms of the set of von Neumann algebra effects). Furthermore, in the paper [6] Gudder and Latrémolière presented an interesting result in which they characterized the sequential product on Hilbert space effect algebras.

Driven by the above mentioned research and results, in the present paper we investigate the operation in question in the setting of general $C^*$-algebras rather than in the particular case of a full operator algebra over a given Hilbert space or in that of a matrix algebra. In the second section of the paper we deal with the algebraic properties of the operation $\circ$, namely, with its commutativity, associativity and distributivity. In the third section we present characterizations of that operation and a few related ones, too.

Throughout this paper $\mathcal{A}$ denotes a unital $C^*$-algebra, $\mathcal{A}_+$ stands for the cone of all positive elements in $\mathcal{A}$ (elements which are self-adjoint and have non-negative spectrum), and $\mathcal{A}_+^{-1}$ denotes the set of all invertible elements in $\mathcal{A}_+$. We equip $\mathcal{A}_+$ with the operation

$$a \circ b = \sqrt{a} b \sqrt{a}, \qquad a, b \in \mathcal{A}_+.$$

As mentioned above and being reflected in the title of the paper, the structure $(\mathcal{A}_+^{-1}, \circ)$ is an important example of so-called K-loops. In fact, on the set $\mathcal{A}_+^{-1}$ the operation $\circ$ provides the most natural and standard K-loop structure. We recall the necessary algebraic notions (we adopt the terminology of Kiechle's book [14]). A set $L$ with a binary operation $\star : L \times L \to L$ is called a quasigroup if the equations $a \star x = b$ and $y \star a = b$ have unique solutions $x, y$ in $L$ for all $a, b \in L$. If the quasigroup $L$ contains an identity element, usually denoted by 1, then $L$ is called a loop. Loops which satisfy the so-called Bol identity

$$a \star (b \star (a \star c)) = (a \star (b \star a)) \star c, \qquad a, b, c \in L \tag{1}$$

are called Bol loops. A K-loop is a Bol loop $L$ with the automorphic inverse property

$$(a \star b)^{-1} = a^{-1} \star b^{-1}, \qquad a, b \in L. \tag{2}$$

In verifying that $\mathcal{A}_+^{-1}$ with the operation $\circ$ really forms a K-loop there is only one non-easy step, namely the unique solvability of the equation $y \circ a = b$ for any $a, b \in \mathcal{A}_+^{-1}$ which is equivalent to the fact that the so-called Riccati equation $ya^{-1}y = b$ has a unique solution $y \in \mathcal{A}_+^{-1}$ for any $a, b \in \mathcal{A}_+^{-1}$. In fact, this unique solution happens to be the geometric mean $a\#b = \sqrt{a}(\sqrt{a}^{-1} b \sqrt{a}^{-1})^{1/2}\sqrt{a}$ of $a$ and $b$ which is the content of the so-called Anderson-Trapp theorem, see [1]. (Observe that in [1] the particular case where $\mathcal{A}$ is the algebra of all bounded linear operators acting on a Hilbert space is considered but by Gelfand-Naimark theorem every abstract $C^*$-algebra is isomorphic to a closed *-subalgebra of the $C^*$-algebra of Hilbert space operators.)

There is another algebraic concept which has turned to be equivalent to the notion of K-loops. It is called gyrogroup (more precisely gyrocommutative gyrogroup), the fundamental theory of which was worked out by Ungar in his book [26]. In fact, along his investigations of the relativistic addition $\oplus$ of velocities defined on the set $\mathbb{R}^3_c = \{v \in \mathbb{R}^3 : |v| < c\}$ he showed that the structure $\mathbb{R}^3_c$ is a non-associative and non-commutative loop with certain characteristic automorphisms, and he gave the name "gyrogroups" to the corresponding abstract structures. In the book [26] he presented the essentials of their theory pointing out that they can provide an adequate algebraic background for Albert Einstein's special theory of relativity as well as for analytic hyperbolic geometry. The fact that the notion of the so-called gyrocommutative gyrogroups is equivalent to that of the K-loops was proved in [25] (in that paper gyrocommutativity is a part of the definition of gyrogroups).

## 2. Algebraic properties of the $\circ$-product

In this section we investigate certain algebraic properties of the product $\circ$ on $\mathcal{A}_+^{-1}$. Namely, we consider its commutativity, associativity and distributivity (with respect to the usual addition $+$). In all cases we find that the corresponding property holds for $\circ$ on $\mathcal{A}_+^{-1}$ if and only if $\mathcal{A}$ is a commutative $C^*$-algebra. We begin with commutativity. For Hilbert space effect algebras the first statement in the result below was obtained in Corollary 2.2 in [7] (in that structure, elements which commute with respect to the operation $\circ$ are called sequentially independent.) The proof was based on the Fuglede-Putnam-Rosenblum theorem. Here we present a short proof which works for any unital $C^*$-algebra and avoids the use of that deep result. We recall the well-known fact that in any unital algebra $\mathcal{R}$, for any pair $a, b \in R$ of elements we have $\sigma(ab)\backslash\{0\} = \sigma(ba)\backslash\{0\}$, where $\sigma(.)$ stands for the spectrum of elements.

**Proposition 1.** *For any $a, b \in \mathcal{A}_+$ we have $a \circ b = b \circ a$ if and only if $ab = ba$. It follows that the operation $\circ$ on $\mathcal{A}_+^{-1}$ is commutative if and only if the algebra $\mathcal{A}$ is commutative.*

*Proof.* The sufficiency part of the statement is trivial. As for the necessity, assume that $a \circ b = b \circ a$. It is obvious that this equality is equivalent to the normality of the element $\sqrt{a}\sqrt{b}$. On the other hand, we have

$$\sigma(\sqrt{a}\sqrt{b}) \cup \{0\} = \sigma(\sqrt[4]{b}\sqrt{a}\sqrt[4]{b}) \cup \{0\} \subset [0, \infty[.$$

Therefore, the spectrum of the normal element $\sqrt{a}\sqrt{b}$ is real which means that it is self-adjoint. Hence $\sqrt{a}, \sqrt{b}$ commute implying that $ab = ba$. $\square$

We continue with associativity. Similarly as above, we shall see that the operation $\circ$ is associative precisely when the algebra $\mathcal{A}$ is commutative. In fact, we investigate weaker forms of associativity that appear in relation with the so-called Moufang loops, see, e.g., 3.1 Theorem in [4]. Namely, we consider the following conditions

(a) $(a \star a) \star b = a \star (a \star b)$ called left alternative identity;
(b) $(a \star b) \star b = a \star (b \star b)$ called right alternative identity;
(c) $a \star (b \star a) = (a \star b) \star a$ called flexible identity.

It is trivial to see that the operation $\circ$ satisfies the left alternative identity, i.e., we have

$$(a \circ a) \circ b = a \circ (a \circ b), \qquad a, b \in \mathcal{A}_+.$$

We are going to show that it satisfies any of the identities $(b), (c)$ only if $\mathcal{A}$ is commutative and hence $(\mathcal{A}_+^{-1}, \circ)$ is a Moufang loop precisely when $\mathcal{A}$ is commutative.

We shall need the following lemma. Throughout the paper by a projection we always mean a self-adjoint idempotent.

**Lemma 2.** *Assume $p, q$ are projections in a unital $C^*$-algebra such that $pqp$ is also a projection. Then $p, q$ necessarily commute.*

*Proof.* From $pqp = (pqp)^2 = pqpqp$ we obtain $pq(1-p)qp = 0$, i.e., $((1-p)qp)^*((1-p)qp) = 0$. This implies $(1-p)qp = 0$ and hence we have $pqp = qp$. Since $pqp$ is self-adjoint, so is $qp$ yielding that $pq = qp$. □

**Proposition 3.** *If the operation $\circ$ satisfies any of the two weak associativity properties*

$$(a \circ b) \circ b = a \circ (b \circ b), \qquad a, b \in \mathcal{A}_+^{-1}, \tag{3}$$

*or*

$$(a \circ b) \circ a = a \circ (b \circ a), \qquad a, b \in \mathcal{A}_+^{-1}, \tag{4}$$

*then the algebra $\mathcal{A}$ is necessarily commutative.*

*Proof.* First observe that if any of the equalities (3), (4) holds on $\mathcal{A}_+^{-1}$, then it holds also on $\mathcal{A}_+$. This follows from the continuity of the operation $\circ$.

Next we show that it is sufficient to prove the statement for von Neumann algebras. To see this, we note that by Gelfand-Naimark theorem we can assume that $\mathcal{A}$ is a closed *-subalgebra of the algebra $B(H)$ of all bounded linear operators acting on a complex Hilbert space $H$ which contains the identity operator $I$. By von Neumann's double commutant theorem the double commutant $\mathcal{A}''$, which is a von Neumann algebra that we denote by $\mathcal{B}$, equals the closure of $\mathcal{A}$ in the strong (or weak) operator topology. By Kaplansky's density theorem the positive part of the unit ball of $\mathcal{A}$ is strongly dense in the positive part of the unit ball of $\mathcal{B}$. It is well known (and easy to check) that the square root operation and the operation of multiplication are strongly continuous on any bounded sets of positive linear operators. It then follows that in either one of the cases (3) and (4) the corresponding associativity equality holds for all $a, b$ in the von Neumann algebra $\mathcal{B}$, too.

Let us first consider the case (3). Pick projections $p, q$ in $\mathcal{B}$. By (3) we obtain

$$\sqrt{pqp}q\sqrt{pqp} = pqp.$$

Multiplying by $\sqrt{pqp}$ from both sides we get

$$(pqp)^3 = pqpqpqp = (pqp)^2.$$

It follows that for every element $\lambda$ of the spectrum of the positive linear operator $pqp$ we have $\lambda^3 = \lambda^2$, i.e., $\lambda$ is either 0 or 1. We deduce that $pqp$ is a projection and by Lemma 2 we obtain that $pq = qp$. Therefore, any two projections in $\mathcal{B}$ commute. Since a von Neumann algebra is generated by its projections (meaning that the algebra equals the norm-closed linear span of the set of its projections), we conclude that $\mathcal{B}$ is commutative which proves the assertion.

We now consider the case (4). Again, pick projections $p, q$ in $\mathcal{B}$. By (4) we obtain

$$\sqrt{pqp}p\sqrt{pqp} = p(qpq)p.$$

Multiplying by $\sqrt{pqp}$ from both sides we get $(pqp)^2 = (pqp)^3$ as above and we can finish the proof in the same way. $\square$

*Remark* 4. In Theorem 3.2 in [7], Gudder and Nagy proved that in the $C^*$-algebra $B(H)$ of all bounded linear operators acting on a complex Hilbert space $H$ the following holds: if for a given pair $a, b \in \mathcal{A}_+$ of elements the equality

$$(a \circ b) \circ c = a \circ (b \circ c) \tag{5}$$

holds for every $c \in \mathcal{A}_+$, then $a, b$ necessarily commute. We remark that the statement holds in any prime $C^*$-algebra, too (a ring $\mathcal{R}$ is called prime if for any $x, y \in \mathcal{R}$, the equality $x\mathcal{R}y = \{0\}$ implies that either $x = 0$ or $y = 0$). Indeed, (5) implies that

$$\sqrt{\sqrt{a}b\sqrt{a}}c\sqrt{\sqrt{a}b\sqrt{a}} = \sqrt{a}\sqrt{b}c\sqrt{b}\sqrt{a}$$

holds for all $c \in \mathcal{A}$. By Theorem 5.1.7 and Proposition 2.2.10 in [2] we either have $\sqrt{\sqrt{a}b\sqrt{a}} = \sqrt{b}\sqrt{a} = 0$ or $\sqrt{\sqrt{a}b\sqrt{a}}, \sqrt{b}\sqrt{a}$ are linearly dependent. In either case we have a scalar $\lambda \in \mathbb{C}$ such that $\lambda\sqrt{\sqrt{a}b\sqrt{a}} = \sqrt{b}\sqrt{a}$. This gives us that $\sqrt{b}\sqrt{a}$ is normal and by Proposition 1 we obtain that $a, b$ necessarily commute.

If $a, b$ in (5) are invertible, then the situation is more simple, we can avoid the use of the deep results in [2] mentioned above and obtain the same conclusion for arbitrary $C^*$-algebras. Indeed, for invertible $a, b$, the equality (6) implies that for $u = \sqrt{\sqrt{a}b\sqrt{a}}\sqrt{a}^{-1}\sqrt{b}^{-1}$ we have $u^*cu = c$. It is apparent that $u^*u = 1$, i.e., $u$ is unitary. It then follows that $cu = uc$. Consequently, $u$ commutes with every element of the $C^*$-algebra $\mathcal{A}$. Since $u^*\sqrt{\sqrt{a}b\sqrt{a}} = \sqrt{b}\sqrt{a}$, we again conclude that $\sqrt{b}\sqrt{a}$ is normal and hence $a, b$ necessarily commute.

In the last result of this section we consider the distributivity of $\circ$ with respect to the addition $+$. Clearly, distributivity in the second variable, i.e., the equality $a \circ (b + c) = a \circ b + a \circ c$ is valid on $\mathcal{A}_+$. As for distributivity in the first variable, we again obtain that it holds only when the underlying algebra $\mathcal{A}$ is commutative.

**Proposition 5.** *If $c \in \mathcal{A}_+^{-1}$ is such that we have*

$$(a + b) \circ c = a \circ c + b \circ c \tag{6}$$

*for all $a, b \in \mathcal{A}_+^{-1}$, then $c$ is in the center of $\mathcal{A}$. In particular, if* (6) *holds for all $a, b, c \in \mathcal{A}_+^{-1}$, then the algebra $\mathcal{A}$ is commutative.*

*Proof.* First observe that using the same argument as in the first part of the proof of Proposition 3, we may assume that $\mathcal{A}$ is a von Neumann algebra and $c \in \mathcal{A}_+$ is such that (6) holds for all $a, b \in \mathcal{A}_+$. Choose mutually orthogonal projections $p, q \in \mathcal{A}$. Then by (6) we have $(p + q)c(p + q) = pcp + qcq$. It follows that $pcq+qcp = 0$. Multiplying by $q$ from the right we obtain $pcq = 0$ whenever $p, q \in \mathcal{A}$ are mutually orthogonal projections. In particular, we have $pc(1 - p) = 0$ which implies $pc = pcp$. Since this latter element is self-adjoint, it follows that $c$ commutes with $p$. Since this holds for every projection in the von Neumann algebra $\mathcal{A}$, we deduce that $c$ is a central element. This completes the proof of the statement. □

*Remark* 6. It follows from the above results that the commutativity, the associativity and the distributivity (with respect to $+$) of the operation $\circ$ defined on $\mathcal{A}_+^{-1}$ (or on $\mathcal{A}_+$) are all equivalent conditions. We believe this is an interesting algebraic phenomenon.

## 3. Characterizations of the $\circ$-product

In this section we present characterizations of the operation $\circ$ on $\mathcal{A}_+^{-1}$. As mentined in the introduction we are motivated by the result Theorem 3.3 in [6] obtained by Gudder and Latrémolière. In fact, that statement is formulated for the Hilbert space effect algebra and it involves rank-one projections and density operators (positive semidefinite linear operators with unite trace). Hence, beside the full operator algebra $B(H)$ over a Hilbert space $H$, also its trace ideal appears there. In contrast to that result, here we would like to present a statement that is valid for the large class of general $C^*$-algebras that carry a faithful trace and we would like to avoid considering particular elements which belong to certain substructures of those algebras (especially meaning the trace ideal in the case of $B(H)$).

As before, let $\mathcal{A}$ be a unital $C^*$-algebra. A positive linear functional $\tau : \mathcal{A} \to \mathbb{C}$ is called a trace if $\tau(ab) = \tau(ba)$ holds for all $a, b$ in $\mathcal{A}$. The trace $\tau$ is said to be faithful if for any $a \in \mathcal{A}$, the equality $\tau(a^*a) = 0$ implies $a = 0$. We note that many of the unital $C^*$-algebras that provide interesting examples for the theory and for its applications do have faithful traces. Fundamental examples are UHF-algebras, finite factors, irrational rotation algebras and so forth.

Now our first characterization reads as follows.

**Theorem 7.** *Assume $\mathcal{A}$ has a faithful trace $\tau$. Let $\bullet$ be a binary operation on $\mathcal{A}_+^{-1}$ with the following properties:*

(ai) *for every pair $a, b$ of elements of $\mathcal{A}_+^{-1}$, the equation $a \bullet x = b$ has a unique solution $x \in \mathcal{A}_+^{-1}$;*

(aii) *$a \bullet 1 = a$ for all $a \in \mathcal{A}_+^{-1}$;*

(aiii) *$a^2 \bullet b = a \bullet (a \bullet b)$ for all $a, b \in \mathcal{A}_+^{-1}$;*

(aiv) *$\tau((a \bullet b)c) = \tau(b(a \bullet c))$ for all $a, b, c \in \mathcal{A}_+^{-1}$.*

*Then we have $a \bullet b = a \circ b = \sqrt{a}b\sqrt{a}$ for any $a, b \in \mathcal{A}_+^{-1}$.*

For the proof we need the following lemma whose proof employs rather standard arguments. By a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$ we mean a bijective linear transformation which satisfies $J(a^2) = J(a)^2$, $J(a^*) = J(a)^*$ for all $a \in \mathcal{A}$.

**Lemma 8.** *Let $\psi : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ be a bijective additive map. Then there is a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$ such that $\psi(x) = \sqrt{\psi(1)}J(x)\sqrt{\psi(1)}$ holds for all $x \in \mathcal{A}_+^{-1}$.*

*Proof.* We first extend $\psi$ to a bijective linear transformation $\Psi : \mathcal{A} \to \mathcal{A}$. For any four elements $x, y, u, v \in \mathcal{A}_+^{-1}$ we define

$$\Psi((x - y) + i(u - v)) = (\psi(x) - \psi(y)) + i(\psi(u) - \psi(v)).$$

It requires only simple elementary considerations to see that $\Psi$ is well-defined. Since every self-adjoint element of $\mathcal{A}$ is the difference of two positive ones and hence also that of two positive invertible ones, we deduce that $\Psi$ maps $\mathcal{A}$ onto $\mathcal{A}$. The additivity of $\Psi$ is clear and the kernel of $\Psi$ is obviously trivial. Therefore, $\Psi$ is a bijective additive map of $\mathcal{A}$.

We show that it is homogeneous, too. Observe that by the additivity of $\psi$ we have that $\psi(rx) = r\psi(x)$ holds for all $x \in \mathcal{A}_+^{-1}$ and for all positive rational number $r$. Pick any positive real number $\lambda$ and positive rationals $r, s$ such that $r < \lambda < s$. By $\psi(\lambda x) = \psi(rx) + \psi((\lambda - r)x)$ we have $r\psi(x) = \psi(rx) \leqslant \psi(\lambda x)$ and in a similar fashion we obtain $\psi(\lambda x) \leqslant \psi(sx) = s\psi(x)$. Letting $r, s$ tend to $\lambda$, it then follows that $\psi(\lambda x) = \lambda\psi(x)$ holds for all $x \in \mathcal{A}_+^{-1}$ and positive real number $\lambda$. Clearly, it implies that $\Psi$ is positive homogeneous, and then one can readily verify that $\Psi$ is real homogeneous and, finally, that it is complex homogeneous, too. Therefore, $\Psi : \mathcal{A} \to \mathcal{A}$ is a linear bijection.

We next assert that $\Psi$ is bounded. Indeed, for each $x \in \mathcal{A}_+^{-1}$ we have $x \leqslant \|x\|1$. For any real number $\epsilon > 0$ we have $(\|x\| + \epsilon)1 = x + ((\|x\| + \epsilon)1 - x)$ implying that

$$\psi(x) \leqslant \psi((\|x\| + \epsilon)1) = (\|x\| + \epsilon)\psi(1).$$

From this we deduce that $\|\psi(x)\| \leqslant M\|x\|$ holds for all $x \in \mathcal{A}_+^{-1}$, where $M = \|\psi(1)\|$. Now, by suitable decomposition of elements of $\mathcal{A}$ as combinations of positive invertibles, it is not difficult to see that $\Psi$ is continuous at zero and hence it is bounded and thus has a bounded inverse, too. Since $\Psi(\mathcal{A}_+^{-1}) = \psi(\mathcal{A}_+^{-1}) = \mathcal{A}_+^{-1}$ and the closure of $\mathcal{A}_+^{-1}$ is $\mathcal{A}_+$, we deduce that $\Psi(\mathcal{A}_+) = \mathcal{A}_+$. This implies that $\Psi$ when restricted onto the self-adjoint part of $\mathcal{A}$ is a linear order-automorphism. Clearly, the transformation $\sqrt{\Psi(1)}^{-1}\Psi(\cdot)\sqrt{\Psi(1)}^{-1}$ is a unital linear order-automorphism on the self-adjoint part of $\mathcal{A}$ and hence, by a famous theorem of Kadison (see Corollary 5 in [13]), it is necessarily a Jordan *-automorphism of $\mathcal{A}$. This concludes the proof of the lemma. $\square$

After this preparation we can present the proof of Theorem 7.

*Proof of Theorem 7.* Select $a \in \mathcal{A}_+^{-1}$ and define $\psi_a(x) = a \bullet x$, $x \in \mathcal{A}_+^{-1}$. By $(ai)$, the map $\psi_a : \mathcal{A}_+^{-1} \to \mathcal{A}_+^{-1}$ is a bijection. We show that it is also additive. To see this, first observe that we can rewrite $(aiv)$ as

$$\tau(\psi_a(x)c) = \tau(x\psi_a(c)), \qquad a, x, c \in \mathcal{A}_+^{-1}. \tag{7}$$

Using this we compute

$$\begin{aligned}\tau(\psi_a(x+y)c) &= \tau((x+y)\psi_a(c))\\ &= \tau(x\psi_a(c)) + \tau(y\psi_a(c)) = \tau((\psi_a(x) + \psi_a(y))c)\end{aligned}$$

which holds for any $x, y, c \in \mathcal{A}_+^{-1}$. By the faithfulness of $\tau$ it follows that $\psi_a(x+y) = \psi_a(x)+\psi_a(y)$, $x, y \in \mathcal{A}_+^{-1}$ yielding the additivity of $\psi_a$. Applying Lemma 8 we have a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$\psi_a(x) = \sqrt{\psi_a(1)}J(x)\sqrt{\psi_a(1)} = \sqrt{a}J(x)\sqrt{a}, \qquad x \in \mathcal{A}_+^{-1},$$

where in the last equality we have used the property $(aii)$. From $(aiii)$ it follows that $a^2 = a^2 \bullet 1 = a \bullet (a \bullet 1) = a \bullet a$. This implies that $a^2 = \psi_a(a) = \sqrt{a}J(a)\sqrt{a}$ from which we infer $a = J(a)$. Since $J$ is a Jordan *-isomorphism, it follows that $\sqrt{a} = J(\sqrt{a})$. It is well known that any Jordan homomorphism respects the Jordan triple product $aba$, see, e.g., [24], 6.3.2 Lemma. Therefore, we have

$$\psi_a(x) = \sqrt{a}J(x)\sqrt{a} = J(\sqrt{a})J(x)J(\sqrt{a}) = J(\sqrt{a}x\sqrt{a}), \qquad x \in \mathcal{A}_+^{-1}. \tag{8}$$

Now, applying (7) we deduce

$$\begin{aligned}\tau((J(\sqrt{a}x\sqrt{a}))c) &= \tau(\psi_a(x)c) = \tau(x\psi_a(c))\\ &= \tau(x\sqrt{a}J(c)\sqrt{a})) = \tau(\sqrt{a}x\sqrt{a}J(c))\end{aligned}$$

for any $x, c \in \mathcal{A}_+^{-1}$. This clearly implies the equality

$$\tau(J(x)c) = \tau(xJ(c)), \qquad x, c \in \mathcal{A}_+^{-1}. \tag{9}$$

In particular, plugging $c = 1$ and referring to $J(1) = 1$, it follows that $J$ is trace preserving. Moreover, using $(aiii)$, the previously obtained equalities (8), (9), the Jordan triple product preserving property and the trace preserving property of $J$ we compute

$$\begin{aligned}\tau((a^2 \bullet x)c) &= \tau((a \bullet (a \bullet x))c) = \tau(\psi_a^2(x)c) = \tau(J^2(axa)c)\\ &= \tau(J(axa)J(c)) = \tau(J(\sqrt{c})J(axa)J(\sqrt{c}))\\ &= \tau(J(\sqrt{c}axa\sqrt{c})) = \tau(\sqrt{c}axa\sqrt{c}) = \tau((axa)c).\end{aligned}$$

By the faithfulness of the trace $\tau$ we obtain $a^2 \bullet x = axa$. Finally, replacing $a$ by $\sqrt{a}$ we complete the proof. $\square$

We remark that, in a way, in the formulation of the above theorem we have followed the formulation of the result Theorem 3.3 in [6]. Namely, just as there, in Theorem 7 the original multiplication and the operation $\bullet$ appear in a mixed way. Apparently, one can find this situation rather confusing and hence we show a possibility to avoid this. The following is a trivial corollary of our former result.

**Corollary 9.** *Assume $\mathcal{A}$ has a faithful trace $\tau$. Let $\bullet$ be a binary operation on $\mathcal{A}_+^{-1}$ with the following properties:*

(bi) *for every pair $a, b$ of elements of $\mathcal{A}_+^{-1}$, the equation $a \bullet x = b$ has a unique solution $x \in \mathcal{A}_+^{-1}$;*
(bii) *$a \bullet 1 = a$ and $a \bullet a = a^2$ for all $a \in \mathcal{A}_+^{-1}$;*
(biii) *$(a \bullet a) \bullet b = a \bullet (a \bullet b)$ for all $a, b \in \mathcal{A}_+^{-1}$;*
(biv) *$\tau((a \bullet b) \bullet c) = \tau(b \bullet (a \bullet c))$ for all $a, b, c \in \mathcal{A}_+^{-1}$;*
(bv) *$\tau(a \bullet b) = \tau(ab)$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Then we have $a \bullet b = a \circ b = \sqrt{a}b\sqrt{a}$ for all $a, b \in \mathcal{A}_+^{-1}$.*

For a result of a somewhat similar spirit characterizing the usual matrix multiplication we refer to the paper [10].

In our next theorem we present an abstract characterization of the standard K-loop operation $\circ$ for all $C^*$-algebras. To this we recall that, by the result (6.8) in [14], a Bol loop $(L, \star)$ is a K-loop if and only if it satisfies the identity

$$a \star ((b \star b) \star a) = (a \star b) \star (a \star b), \qquad a, b \in L \tag{10}$$

which is sometimes called Bruck identity, see, e.g., p. 763 in [17]. The following result shows that the only left loop operation $\bullet$ on $\mathcal{A}_+^{-1}$ with unit 1 and $a \bullet a = a \circ a$, $a \in \mathcal{A}_+^{-1}$ which is additive in its second variable and satisfies the identity (10) is the operation $\circ$.

**Theorem 10.** *Let $\bullet$ be a binary operation on $\mathcal{A}_+^{-1}$ with the following properties:*

(ci) *for any $a \in \mathcal{A}_+^{-1}$, the map $x \mapsto a \bullet x$ is bijective and additive on $\mathcal{A}_+^{-1}$;*
(cii) *$a \bullet 1 = a$ and $a \bullet a = a^2$ for all $a \in \mathcal{A}_+^{-1}$;*
(ciii) *$a \bullet ((b \bullet b) \bullet a) = (a \bullet b) \bullet (a \bullet b)$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Then we have $a \bullet b = a \circ b = \sqrt{a}b\sqrt{a}$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Proof.* Select $a, b \in \mathcal{A}_+^{-1}$. By ($ci$), Lemma 8 and ($cii$) we have a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$a \bullet x = \sqrt{a}J(x)\sqrt{a}, \qquad x \in \mathcal{A}_+^{-1}.$$

By ($cii$) we also have $J(a) = a$ which implies $\sqrt{a} = J(\sqrt{a})$. Since $J$, as any Jordan automorphism, preserves the Jordan triple product we then obtain

$$a \bullet x = \sqrt{a}J(x)\sqrt{a} = J(\sqrt{a}x\sqrt{a}).$$

Similarly, to $b^2$ there corresponds a Jordan *-automorphism $J'$ of $\mathcal{A}$ such that

$$b^2 \bullet x = J'(bxb), \qquad x \in \mathcal{A}_+^{-1}$$

holds. Now we compute

$$a \bullet ((b \bullet b) \bullet a) = a \bullet (b^2 \bullet a) = J(\sqrt{a}J'(bab)\sqrt{a})$$

and

$$(a \bullet b) \bullet (a \bullet b) = (a \bullet b)^2 = (J(\sqrt{a}b\sqrt{a}))^2 = J((\sqrt{a}b\sqrt{a})^2) = J(\sqrt{a}bab\sqrt{a}).$$

By ($ciii$) and the injectivity of $J$ it follows that $\sqrt{a}J'(bab)\sqrt{a} = \sqrt{a}bab\sqrt{a}$ which implies

$$b^2 \bullet a = J'(bab) = bab, \qquad a, b \in \mathcal{A}_+^{-1}.$$

Replacing $b$ by $\sqrt{b}$ we complete the proof. $\square$

*Remark* 11. We remark that in the above result we cannot omit the assumption on additivity in the second variable. Indeed, one can easily check that the operation $(ab^2a)^{1/2}$ has all the listed properties ($ci$) $-$ ($ciii$) except for the additivity in the variable $b$.

We also mention that in the literature there is some ambiguity in the use of the notion of Bruck identity. For example, in the papers [23], [25] the authors use this expression for the identity

$$a \star (b \star (b \star a)) = (a \star b) \star (a \star b). \tag{11}$$

Observe that the conclusion in Theorem 10 is no longer valid if we replace ($ciii$) by (11). Indeed, for any $a \in \mathcal{A}_+^{-1}$ choose a symmetry (self-adjoint unitary) $u_a$ which commutes with $a$ and define $a \bullet b = u_a\sqrt{a}b\sqrt{a}u_a$. One can readily verify that this operation satisfies the conditions ($ci$), ($cii$) in Theorem 10 and also the identity (11) but not necessarily equals $\circ$.

In the next result we present a remarkable strengthening of Theorem 10 for the particular $C^*$-algebra $B(H)$ of all bounded linear operators acting on a Hilbert space $H$. Namely, we drop the additivity assumption in ($ci$) and replace it by the weaker assumption of order preservation and, instead of the Bruck identity ($ciii$), we assume a similar equality but only in norm. We recall that for self-adjoint linear operators the usual order $\leqslant$ is defined as follows. For any pair $A, B$ of such operators on $H$ we write $A \leqslant B$ if and only if $\langle Ax, x\rangle \leqslant \langle Bx, x\rangle$ holds for all $x \in H$.

**Theorem 12.** *Let $H$ be a complex Hilbert space and consider a binary operation $\bullet$ on the set $B(H)_+^{-1}$ of all invertible positive linear operators on $H$ with the following properties:*

(di) *for any $A \in B(H)_+^{-1}$, the map $X \mapsto A \bullet X$ is bijective on $B(H)_+^{-1}$ and preserves the usual order in both directions (meaning that for any $X, Y \in B(H)_+^{-1}$ we have $A \bullet X \leqslant A \bullet Y$ if and only if $X \leqslant Y$);*

(dii) *$A \bullet I = A$ and $A \bullet A = A^2$ for all $A \in B(H)_+^{-1}$;*

(diii) *$\|A \bullet ((B \bullet B) \bullet A)\| = \|(A \bullet B) \bullet (A \bullet B)\|$ for all $A, B \in B(H)_+^{-1}$.*

*Then $\bullet$ coincides with the standard K-loop operation $\circ$.*

*Proof.* Pick $A \in B(H)_+^{-1}$. The map $\psi_A : B(H)_+^{-1} \to B(H)_+^{-1}$ defined by $\psi_A(X) = A \bullet X$, $X \in B(H)_+^{-1}$ is an order-automorphism of $B(H)_+^{-1}$ with respect to the usual order $\leqslant$. The structure of such maps have been determined in the paper [22] of the second author. Theorem 1 in [22] tells us that there is an invertible bounded either linear or conjugate-linear operator $T$ on $H$ such that

$$\psi_A(X) = A \bullet X = TXT^*, \qquad X \in \mathcal{A}_+^{-1}.$$

Using $(dii)$ we obtain $TT^* = A$ which implies that $|T^*| = \sqrt{A}$. By polar decomposition $T^* = U\sqrt{A}$ holds with a unitary or antiunitary operator $U$ on $H$. It follows

$$A \bullet X = \sqrt{A}U^*XU\sqrt{A}.$$

Using the second part of $(dii)$ we have $A^2 = \sqrt{A}U^*AU\sqrt{A}$ from which we deduce $A = U^*AU$. It follows that $U$ commutes with $A$ and therefore with $\sqrt{A}$, too. We then infer

$$\|A \bullet X\| = \|U^*\sqrt{A}X\sqrt{A}U\| = \|\sqrt{A}X\sqrt{A}\|, \qquad X \in B(H)_+^{-1}.$$

Pick $B \in B(H)_+^{-1}$. By the argument above we have a unitary or antiunitary operator $V$ on $H$ such that

$$B^2 \bullet X = BV^*XVB, \qquad X \in B(H)_+^{-1}$$

and $V$ commutes with $B$. Using the information collected above we calculate both sides of the equality $\|A \bullet (B^2 \bullet A)\| = \|(A \bullet B)\|^2$ and obtain the following equation

$$\|\sqrt{A}BV^*AVB\sqrt{A}\| = \|\sqrt{A}B\sqrt{A}\|^2.$$

Since

$$\|\sqrt{A}BV^*AVB\sqrt{A}\| = \|(\sqrt{A}VB\sqrt{A})^*(\sqrt{A}VB\sqrt{A})\| = \|\sqrt{A}VB\sqrt{A}\|^2,$$

we obtain that

$$\|\sqrt{A}B\sqrt{A}\| = \|\sqrt{A}VB\sqrt{A}\| \tag{12}$$

holds for every $A, B \in B(H)_+^{-1}$. Fixing $B$ for a moment, it follows by continuity that the same equality holds for every positive semidefinite linear operator $A$ on $H$ and hence, in particular, for any rank one projection

$A = x \otimes x$, where $x$ is an arbitrary unit vector in $H$. The equation (12) implies

$$|\langle VBx, x\rangle| = \langle Bx, x\rangle$$

holds for all unit vectors $x \in H$. We know that $B$ and hence $\sqrt{B}$ commutes with $V$. We infer that

$$|\langle V\sqrt{B}x, \sqrt{B}x\rangle| = |\langle VBx, x\rangle| = \langle Bx, x\rangle = \langle \sqrt{B}x, \sqrt{B}x\rangle$$

is valid for every unit vector $x \in H$ implying that $|\langle Vy, y\rangle| = \langle y, y\rangle$, $y \in H$. It follows that there is equality in the Cauchy-Schwarz inequality

$$|\langle Vy, y\rangle| \leqslant \|Vy\|\|y\| = \|y\|^2.$$

This implies that $Vy$ is a scalar multiple of $y$ for every $y \in H$. It is a folk result whose proof requires only elementary linear algebraic arguments that this property ensures that $V$ is a scalar multiple of the identity. Since $V$ is an isometry, the scalar must be of modulus 1. Therefore, we have

$$B^2 \bullet A = BV^*AVB = BAB$$

for any $A, B \in B(H)_+^{-1}$ and the proof can be completed readily. □

Concerning the above proof we mention that other applications of the structure of order-automorphisms of $B(H)_+^{-1}$ can be found in [22]. To conclude the paper we mention that one can use the results above to characterize some other operations on the set of all positive invertible elements in a $C^*$-algebra. For example, we present two corresponding results concerning the Jordan triple product $aba$ and the inverted Jordan triple product $ab^{-1}a$. The former product plays an important role in ring theory but it also has applications in the algebraic background of infinite dimensional holomorphy, in the theory of triple systems, etc. The inverted Jordan triple product has appeared in our recent papers in which we have extended the famous Mazur-Ulam theorem for the setting of noncommutative metric groups and related structures. See [8] and also [9] where the general results obtained in [8] have been applied to descriptions of norm-isometries on unitary groups and of so-called Thompson isometries on the set of positive invertible elements of $C^*$-algebras.

**Proposition 13.** *Let $\bullet$ be a binary operation on $\mathcal{A}_+^{-1}$ with the following properties:*

(ei) *for any $a \in \mathcal{A}_+^{-1}$, the map $x \mapsto a \bullet x$ is bijective and additive on $\mathcal{A}_+^{-1}$;*
(eii) *$a \bullet 1 = a^2$, $a \bullet a = a^3$ for all $a \in \mathcal{A}_+^{-1}$;*
(eiii) *$a \bullet (b \bullet a^2) = (a \bullet b)^2$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Then we have $a \bullet b = aba$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Proof.* One can follow an argument very similar to what we have used in the poof of Theorem 10. □

As already mentioned above, our second additional characterization concerns the operation $ab^{-1}a$ that we called inverted Jordan triple product in [8]. We have already pointed out that this operation plays an important role in non-commutative generalizations of the Mazur-Ulam theorem (which asserts the affinity of surjective distance preserving maps on normed real-linear spaces). Moreover, this operation on $\mathcal{A}_+^{-1}$ provides us with a fundamental example of so-called symmetric sets which appear (under different names) in several abstract algebraic theories, see, e.g., [19], [17]. Following the former paper we define a symmetric set as a set $X$ equipped with a binary operation $\star$ that satisfies

(fi) $a \star a = a$;
(fii) $a \star (a \star b) = b$;
(fiii) $a \star (b \star c) = (a \star b) \star (a \star c)$

for all $a, b, c \in X$. It is apparent that by $(fii)$ the equation $a \star x = b$ has a unique solution $x \in X$ for every $a, b \in X$. Assume further that $X$ is a subset of a group, $1 \in X$ and $a \star 1 = a^2$ holds for all $a \in X$. Then it is easy to verify that $a \star (b \star a^2) = (a \star b)^2$ holds for all $a, b \in X$. We now conclude our paper with the following characterization of the operation $ab^{-1}a$ on $\mathcal{A}_+^{-1}$.

**Proposition 14.** *Let $\bullet$ be a binary operation on $\mathcal{A}_+^{-1}$ with the following properties:*

(gi) *for any $a \in \mathcal{A}_+^{-1}$, the map $x \mapsto a \bullet x^{-1}$ is bijective and additive on $\mathcal{A}_+^{-1}$;*
(gii) *$a \bullet 1 = a^2$, $a \bullet a = a$ for all $a \in \mathcal{A}_+^{-1}$;*
(giii) *$a \bullet (b \bullet a^2) = (a \bullet b)^2$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Then we have $a \bullet b = ab^{-1}a$ for all $a, b \in \mathcal{A}_+^{-1}$.*

*Proof.* Just as above, one can follow an argument very similar to what we have used in the poof of Theorem 10. We point out that here we need to use the additional fact that Jordan automorphisms of algebras preserve the inverse operation, i.e, $J(a^{-1}) = J(a)^{-1}$ holds for any invertible element $a$ from the domain of $J$. $\square$

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